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The Double Cover of the Real Symplectic Group and a Theme from Feynman's Quantum Mechanics

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Abstract. We present a direct connection between the 2-cocycle defining the double cover of the real symplectic group and a Feynman path integral describing the time evolution of a quantum mechanical system.

Mathematics Subject Classification: Primary 05C38, 15A15; Secondary 05A15, 15A18

1. Introduction

Already in the 1978 monograph *Lagrangian Analysis and Quantum Mechanics* [12], written by Jean Leray in response to a question posed to him by V. I. Arnol'd eleven years before, a bridge is built between the theory of the real metaplectic group of André Weil (cf. [21]), or the double cover of the real symplectic group, and the formalism surrounding the Maslov index. For our purposes, however, we turn to the 1975 paper [20] by Jean-Marie Souriau, presenting, as its title indicates, an explicit construction of the Maslov index, which, again for our purposes, we shall term the Arnol'd-Leray-Maslov index. Souriau's work is connected to that of Leray, more specifically to the latter's 1973 communication [11], but places its stresses elsewhere: while Leray addresses the abstract Fourier analysis that goes into the definition of the real metaplectic group in a novel way, distinct form the approach chosen by Weil (and before him, D. Shale, in the context of physics), Souriau builds on the according connections with symplectic and Lagrangian analysis to provide a formula directly involving the Arnol'd-Leray-Maslov index and the time evolution of a simple (or single) quantum mechanical system, in other words, a unitary operator involving a Feynman path integral equipped with a certain quadratic (or Gaussian) phase function.

Maslov's index, in its first incarnation, was originally presented in 1965, by V. P. Maslov, in a pair of monographs [14][15] one of which directly addressed the W(entzel-)K(ramers-)B(rillouin) method in quantum mechanics; he presented his discussion in the context of phase-integral methods in physics: a very telling circumstance. It was Arnol'd, however, in his famous paper [1] revealingly titled "[A] Characteristic class entering in[to] quantization conditions," who explicated the connections with symplectic geometry alluded to above, a theme in turn taken up by Leray in [12]. It therefore is indeed proper that the index in question be termed the Arnol'd-Leray-Maslov index. However, there is a more prosaic reason to do this, namely, to distinguish it from another index which, while related to it, is not identical with it. In what follows this index will be referred to as the Kashiwara-Maslov index, to signify that its definition goes back to Masaki Kashiwara [7] and that it was G. Lion and M. Vergne, in [13], who introduced the term Maslov index in connection with their analysis of the projective representation of the symplectic group that has come to be called the Weil representation.

Despite this ambiguity, and as already noted, these two types of Maslov indices are closely related and this theme (among others) was addresses very carefully in the important 1994 article, "On the Maslov index" [2], authored by S. Cappell, R. Lee, and E. Y. Miller, in the context of a sweeping and exhaustive comparison of half a dozen equivalent but separate definitions of a Maslov index and an explication of their relationships to other indices including, besides the Kashiwara index, Wall's index and indices of Atyiah-Singer type. We note in this connection that already in [1] Arnol'd addresses (in the paper's last section) a "connection between the Maslov and Morse indices."

Going over, next, to the viewpoint provided by Lion-Vergne in [13], i.e. that of 2-dimensional group cohomology *per se*, the salient point for our objectives is that they establish a means whereby to relate the projective Weil representation of the symplectic group to the Kashiwara-Maslov index. In point of fact this is done in the form of explicit formulas so that the 2-cocycle defining the metaplectic group, which is to say the 2-cocycle that arises due to the projectivity of the Weil representation, is directly expressible in terms of the Kashiiwara-Maslov index. Therefore, putting together what Souriau did in [20] with Lion-Vergne's work, and using the appropriate traslations offered by [16][17], we can derive a direct and explicit realtionship, at least in the present real case, between the aforementioned 2-cocycle, which we will term the Weil 2-cocycle, and what amounts to a unitary operator in the context of quantum mechanics expressed in terms of Feynman's yoga of path integrals. This explicit relationship is the *raison d'être* of the present article.

This having been said, the work we present here is properly speaking a prelude to a number theoretic undertaking aiming at an old open question; our greater aim is to approach this overarching theme in two phases, the first of which is projected for the immediate future, while the second is of course more ephemeral. The open question goes back to 1923 and is due to Erich Hecke who posed it at the end of his *Vorlesungen über die Theorie der Algebraischen Zahlen* [5] and consists in the challenge to craft a generalization to higher reciprocity laws of his Fourier-analytic proof of quadratic reciprocity given in the preceding pages. This entire matter was reformulated in the 1960's by Weil, in [21], who recast Hecke's proof in terms of the behavior of the double cover of the lowest-dimensional symplectic group (identifiable with $SL(2)$) over the ring of adèles of the number field in question: 2-Hilbert(-Hasse) reciprocity follows from the fact that this double cover, equivalently its defining adèlic 2-cocycle, is split on the subgroup of rational points. This latter result in turn follows from fact that a certain natural adèlic linear functional, the Weil Θ-functional, is invariant under the action of the indicated rational points through the agency of the Weil representation. Subsequent to this work on Weil's part, Tomio Kubota established in [10] that n-Hilbert reciprocity follows as a consequence of having the n-fold cover of $SL(2)$ over the adèles split on the rational points; however, as of the date of this writing the proof of this splitting in fact invokes n-Hilbert reciprocity, so this is precisely where the battle should be joined if Hecke's challenge is to be met.

Accordingly, the second and more distant phase of of our projected future investigation conserns this sufficient condition identified by Kubota: we seek to approach this matter not with Fourier analysis directly, as Hecke and Weil brought to bear on the quadratic case, but ultimately with the formalism of Feynman path integrals and, more generally, oscillatory integrals in the sense of Hörmander [6]. With this goal in mind, the first and more proximate phase of our projected strategy is to carry out what we do in the pages that follow for all places of the underlying number field, specifically the non-archimedean ones, and for the corresponding adèles (at least in the lowest dimensional case, which suffices for reciprocity laws). And so, in the present paper, we develop the indicated line in the paradigmatic real case. Thus, this work can be regarded as the initial step, i.e. laying the foundation, for a more ramified attempt to introduce a new methodology into analytic number theory as it pertains to the analytic proof of reciprocity laws for number fields.

2. The arrangement of the paper

In light of our preceding remarks, in the Introduction, it is natural to split our presentation into four parts, or four sections. We begin with a lay out of the theory and fundamental facts surrounding what we will here refer to a "the" metaplectic group (over $\mathbb{R}$). There is some ambiguity to this term and it is perhaps apposite to address this matter, if only very briefly. The adjective itself goes back to André Weil himself who introduced it in 1964 in the context of his seminal paper [21]; in a private communication [22] to the present author dating to a quarter of a century later Weil also used this nomenclature in connection with the object of present interest to us, viz. the double cover of the symplectic group. To be precise, however, and as was already hinted, with his greater goal being a unitary representation theoretic explication of C. L. Siegel's analytic theory of quadratic forms, Weil's focus fell not only on the real symplectic group (of lowest dimension) but on the symplectic groups over all the local fileds obtained as completions of the base ($\mathbb{A}$-)field at its primes, archimedean or not, and then over the attendant ring of adèles over this base field. However we train our attention exclusively on the real case in the pages that follow.

But this is only part of the story as far as the term "metaplectic" is concerned. As we also noted above, Weil's pioneering work on quadratic reciprocity in [21] was taken up soon afterward by Kubota who first replaced the symplectic group by $SL(2)$ in [9], and provided a 2-cocycle for the all-important double cover directly in terms of the 2-Hilbert symbol, and then, in [10], addressed the question of higher Hilbert reciprocity laws: n-Hilbert reciprocity was dealt with in terms of n-fold covers of $SL(2)$, locally and adèlically. The latter are today also often referred to as metaplectic groups, in a different sense than Weil's original one. The hugely important paper, "Metaplectic forms' [8], by D. Kazhdan and S. J. Patterson, doubtless contributed profoundly to the popularity of this usage which is probably the prevailing one in number theory.

Even so, although Weil's original, more restrictive, terminology is not favored by arithmeticians, it is the prevailing language for other workers in the field, e.g., mathematical physicists. This is particularly apt since the subject's prehistory involves a paper [19] by David Shale, dealing with particles satisfying Bose-Einstein statistics, in which the author, building on work of I. E. Segal [18], introduces a projective representation of a symplectic group which is a prototype for the object now termed the oscillator representation, the Segal-Shale-Weil representation, or, probably most commonly, just the Weil representation. The latter nomenclature attests to the great significance and influence of Weil's work in this area: his 1964 paper [21] addressed above is the cornerstone of all arithmetical activity in both of the areas mentioned, i.e. the study of metaplectic groups as such and the accompanying study of the aforementioned projective representation.

So, come what may, it is proper for us to start with a discussion of this arithmetical and unitary representation theoretic material, and our subsequent section deals with the relevant material concerning the real symplectic group as located within the metaplectic group, the latter being a double cover of the former.

Thereafter we turn to the natural transitional themes, given our objective, namely, the Arnol'd-Leray-Maslov (and Souriau) index, the Kashiwara(-Maslov) index, and their interconnections. The former is the focus of both the classic monograph [12] by Jean Leray himself and the paper [20] by Jean-Marie Souriau which is so important to our purposes. The latter is dealt with in great detail by Gerard Lion and Michele Vergne in [13]; the relationship between them is treated in [2] by Cappell, Lee, and Miller.

Having brought Souriau into the game we are now in a position to bring in his analysis, presented at the end of [20], dealing with forging a direct connection between the Arnol'd-Leray-Maslov index and a Feynman integral. Our focus falls on a relation which provides the time evolution of a single QM system attached to, or equipped with, a quadratic phase function and involves not only the indicated Feynman integral but also a multiplier expression involving the Arnol'd-Leray-Maslov index. We note, too, and discuss below, that this relation appears to make another appearance in a much more recent work, namely [17] by Robbin and Salamon. In their precursor [16] to this article they note explicitly that "[their] treatment is motivated by the formal similarity between Feynman path integrals and the Fourier integrals of Hörmander," and this observation also has bearing on what we are doing in the present paper: Hörmander's theory of oscillatory integrals (cf. [6]) provides the proper larger perspective on our undertaking.

Finally, with the aforementioned relation in place we are in a position to assemble the pieces we crafted, or, rather, developed, in the aforementioned three sections and accordingly present our central result in the closing section of the paper: a direct connection between the real metaplectic group's 2-cocycle and a Feynman path integral for a certain simple QM system's time evolution. We finish by addressing, briefly, the connection between our result and the work of Robbin-Salamon [16][17] mentioned earlier.

3. The real metaplectic group

Following [13], let V be a $2n$-dimensional real vector space and let B be a non-degenerate skew-symmetric quadratic form on V. We have that V is a (real) symplectic space with respect to B if V admits a basis $\{P_i; Q_j\}_{1\leq i,j\leq n}$ such that, for all i, j,

$$B(P_i, P_j) = 0 = B(Q_i, Q_j) \; ; \; B(P_i, Q_j) = \delta_{ij} = -B(Q_j, P_i) \tag{3.1}$$

with B mapping bilinearly into $\mathbb{R}$ and δ_{ij} the Kronecker delta. Introduce a formal symbol E and define the Heisenberg Lie algebra

$$\mathfrak{N} := V \oplus \mathbb{R}E = (\bigoplus_{i=1}^{n} \mathbb{R}P_i) \oplus (\bigoplus_{j=1}^{n} \mathbb{R}Q_j) \oplus \mathbb{R}E \tag{3.2}$$

by stipulating that the attendant Lie bracket should be defined by

$$[\mathfrak{N}, E] = 0 \; ; \; [x, y] = B(x, y)E \tag{3.3}$$

for any $x, y \in V$. Then the Heisenberg group for this data is

$$Heis(V; B) := \exp(\mathfrak{N}) \tag{3.4}$$

and we have, for the indicated group law,

$$\exp(x + tE)\exp(y + tE) = \exp(x + y + \frac{1}{2}B(x, y)E). \tag{3.5}$$

An n-dimensional subspace l of V is a Lagrangian plane if it is self-dual with respect to B, i.e. $l^{\perp} = l$ where

$$l^{\perp} = \{x \in V \mid \forall y \in l, B(x,y) = 0\}. \tag{3.6}$$

The collection $\mathfrak{Lag}(V)$ of all such Lagrangian planes in V (where we take B for granted once and for all) actually carries the structure of an algebraic variety, as we shall have occasion to consider in somewhat greater detail later; it is called the Lagrangian Grassmannian for the data (V, B). It is standard fare that $V = (\bigoplus_{i=1}^{n} \mathbb{R}P_i) \oplus (\bigoplus_{j=1}^{n} \mathbb{R}Q_j)$ is indeed a decomposition into so-called transverse Lagrangian planes: if $l = \bigoplus_{i=1}^{n} \mathbb{R}P_i$ and $l' = \bigoplus_{j=1}^{n} \mathbb{R}Q_j$ then $l = l^{\perp}$ and $l' = (l')^{\perp}$; additionally, $l \cap l' = (0)$. Thus, the decomposition we started with is by no means exotic, but it is obviously very important: we fix this decomposition

$$V = l \oplus l' \tag{3.7}$$

and take L to be the following subgroup of $\mathfrak{N}$

$$L = \exp(l \oplus \mathbb{R}E). \tag{3.8}$$

Consider the character $\chi_L \in L^{\wedge} = Hom(L, \mathbb{C}_1^{\times})$ defined by

$$\chi_L : \exp(x + tE) \longmapsto e^{2\pi i t}. \tag{3.9}$$

Abusing language a little (in keeping with what Lion-Vergne do in [13]) by also writing χ_L for $\chi_L \circ \exp$, we can realize χ_L as acting in the algebra $\mathcal{U}(\mathcal{H}(l))$ of unitary operators on the natural Hilbert space

$$\mathcal{H}(l) := \{\varphi \in L^2(\mathfrak{N}) \mid \forall y \in l, \forall x \in \mathfrak{N}, \varphi(yx) = \chi_L^{-1}(y)\varphi(x)\} \tag{3.10}$$

by means of the rule

$$\chi_L(\exp(x + tE)) = e^{2\pi i t} \cdot id_{\mathcal{H}(l)}; \tag{3.11}$$

in other words, $\chi_L : L \to \mathcal{U}(\mathcal{H}(l))$ *via*

$$\exp(x + tE) \longmapsto [\varphi \mapsto e^{2\pi i t} \cdot \varphi]. \tag{3.12}$$

But this says that χ_L is a central character, given that $\mathcal{Z}(\mathfrak{N}) = \exp \mathbb{R}E$, and so we may invoke the theorem of Stone and von Neumann to great advantage: the Heisenberg group is, by our earlier definition (3.4), $\exp(\mathfrak{N}) =: Heis(V, B)$ and then the Schrödinger representation of this Heisenberg group,

$$Ind_L^{\exp(\mathfrak{N})}(\chi_L) : \exp(\mathfrak{N}) \to \mathcal{U}(\mathcal{H}(l)) \tag{3.13}$$

is irreducible; therefore, if

$$\varrho : Heis(V, B) \to \mathcal{U}(\mathfrak{H}) \tag{3.14}$$

is any other unitary representation of $Heis(V, B)$ such that $\varrho|_{\mathcal{Z}(\exp \mathfrak{N})} = \chi_L$ (where we also have that $\mathcal{Z}(\exp \mathfrak{N}) = \mathbb{R}E$ (loc.cit.)), then ϱ and the Schrödinger representation are unitarily equivalent. In other words, for all $x \in \exp(\mathfrak{N})$ we have

$$\varrho(x) = U^{-1} \circ Ind_L^{\exp(\mathfrak{N})}(\chi_L) \circ U \tag{3.15}$$

for some unitary mapping $U : \mathfrak{H} \xrightarrow{\sim} \mathcal{H}(l)$.

It is evident that the Schrödinger representation depends entirely on the choice of Lagrangian plane l which motivates the notation of $W(l)$ for this irreducible representation, as given by Lion-Vergne whose discussion, in [13], we are following rather closely, of course. With this abbreviation in place we obtain that if l_1 and l_2 are any two Lagrangian planes in V, i.e. $l_1, l_2 \in \mathfrak{Lag}(V)$, the preceding relation (3.15) yields the intertwining

$$W(l_1)(x) = \mathsf{FT}_{1,2} \circ W(l_2)(x) \circ \mathsf{FT}_{2,1} \tag{3.16}$$

for all $x \in \exp(\mathfrak{N})$; here $\mathsf{FT}_{2,1} : \mathcal{H}(l_1) \to \mathcal{H}(l_2)$ is a (partial) Fourier transform in a most natural way (cf. p.34 of *loc.cit.*) and $\mathsf{FT}_{1,2} = \mathsf{FT}_{2,1}^{-1}$.

Next, in light of the preceding we can define the symplectic group for the data $V, B)$ as

(3.17)
$$Sp(V,B) = Sp(2n,\mathbb{R}) = \{\sigma \in GL(2n,\mathbb{R}) \mid \forall x, y \in V, B(x^\sigma, y^\sigma) = B(x,y)\}$$

and provide it with an action on $\exp(\mathfrak{N})$ *via*

$$\sigma : \exp(x + tE) \mapsto \exp(x^\sigma + tE) \tag{3.18}$$

for which, obviously, $\sigma|_{\mathcal{Z}(\exp\mathfrak{N})} = id|_{\mathcal{Z}(\exp\mathfrak{N})}$. Accordingly, $(W(l), \mathcal{H}(l))$ is a unitary $\exp(\mathfrak{N})$-module and if we write $W^\sigma(l) : \exp(\mathfrak{N}) \to \mathcal{U}(\mathcal{H}(l))$ for the data $W(l^\sigma) : \mathcal{H}(l) \to \mathcal{H}(l)$, then, noting that $W^\sigma(l)|_{\mathcal{Z}(\exp\mathfrak{N})} = \chi_L$, it follows from the foregoing that for all $\sigma \in Sp(2n,\mathbb{R})$ we get that

$$W(l)(x) = \mathsf{FT}_\sigma^{-1} \circ W^\sigma(l)(x) \circ \mathsf{FT}_\sigma \tag{3.19}$$

where, obviously, $\mathsf{FT}_\sigma : \mathcal{H}(l) \to \mathcal{H}(l)$, realizing $\mathcal{H}(l)$ first as the representation space for $W(l)$, then as the representation space for $W^\sigma(l)$.

But now Hilbert-Schmidt theory (*loc.cit.*, p.21 ff.) provides that a bounded unitary operator on $\mathcal{H}(l)$ commuting with all the $W(l)(x)$ is necessarily a scalar, whence it follows from a straightforward manipulation of the commutative diagrams expressing the indicated intertwinings of the type (3.19) that for all $\sigma_1, \sigma_2 \in Sp(2n,\mathbb{R})$, we have $\mathsf{FT}_{\sigma_1} \circ \mathsf{FT}_{\sigma_2} \circ \mathsf{FT}_{\sigma_1\sigma_2}^{-1} \in \mathbb{C}^\times$; it then follows from unitarity that $|\mathsf{FT}_{\sigma_1} \circ \mathsf{FT}_{\sigma_2} \circ \mathsf{FT}_{\sigma_1\sigma_2}^{-1}| = 1$.

Furthermore, if we express the relationship between the given Fourier transforms in the form

$$\mathsf{FT}_{\sigma_1\sigma_2} = c(\sigma_1,\sigma_2)\mathsf{FT}_{\sigma_1} \circ \mathsf{FT}_{\sigma_2} \tag{3.20}$$

then the usual manouvres with associativity suffice to establish that

$$c : Sp(2n,\mathbb{R}) \times Sp(2n,\mathbb{R}) \to \mathbb{C}_1^\times, \tag{3.21}$$

as presented above, gives a 2-cocycle

$$c \in H^2(Sp(2n,\mathbb{R}), \mathbb{C}_1^\times). \tag{3.22}$$

In other words, FT yields a projective unitary representation with c as its associated factor set. It is this projective representation that is called the Segal-Shale-Weil representation, i.e. the Weil representation for short, or the oscillator representation, as we mentioned above.

It is a classic result of André Weil [21] that this factor set c as given by (3.20) actually takes its values in the group $\{1,-1\}$, traditionally written as μ_2, so that (3.22) can be strengthened to $c \in H^2(Sp(2n,\mathbb{R}),\mu_2)$; this in turn is tautologically equivalent to the fact that a short exact sequence

$$1 \to \mu_2 \to Sp(2n,\mathbb{R}) \times_c \mu_2 \to Sp(2n,\mathbb{R}) \to 1 \tag{3.23}$$

is in place, defining the central object $Sp(2n,\mathbb{R})\times_c\mu_2$ as the metaplectic group discussed above, its group law being twisted by c: if $\sigma_1,\sigma_2 \in Sp(2n,\mathbb{R})$ and $\xi_1,\xi_2 \in \mu_2$, then

$$(\sigma_1,\xi_1)(\sigma_2,\xi_2) = (\sigma_1\sigma_2, c(\sigma_1,\sigma_2)\xi_1\xi_2). \tag{3.24}$$

4. Maslov indices

There are two indices which figure in what we are doing here, namely, the index that Lion and Vergne term the Maslov index, but we will refer to as the Kashiwara-Maslov index, and the index introduced by V. P. Maslov in [14] and developed by, among others, V. I. Arnol'd [1], Jean Leray [12], and J-M. Souriau [20], the Arnol'd-Leray-Maslov index. The Kashiwara-Maslov index is tied directly to $c \in H^2(Sp(2n,\mathbb{R}),\mu_2)$ by means of developments presented in [13]; this index can be tied to the Arnol'd-Leray-Maslov index by means of a set of results due to Cappell-Lee-Miller in [2]. In the context of Souriau's treatment in [20] the latter index appears in an expression for the time-evolution of a single quantum mechanical system, i.e. a one-parameter Lie group of unitary operators in a natural Hilbert space of states. This quantum mechanical relation is expressed in Feynman's language for quantum mechanics, as we shall see in the next section; in this section, however, we discuss the two flavors of Maslov index that enter into our analysis. We continue to follow [13] closely.

Given any triple of pairwise transverse Lagrangian planes l_1,l_2,l_3 in V, i.e. $l_1,l_2,l_3 \in \mathfrak{Lag}(V)$, consider the quadratic form $q_B : l_1 \oplus l_2 \oplus l_3 \to \mathbb{R}$ defined as

$$q_B(x_1+x_2+x_3) = B(x_1,x_2) + B(x_2,x_3) + B(x_3,x_1) \tag{4.1}$$

and then set

$$\tau(l_1,l_2,l_3) := sgn(q_B) \tag{4.2}$$

where sgn stands for the signature map. In τ we have defined the Kashiwara-Maslov index, the first index in this game. The hypothesis of transversality is weakened in the context of another theorem in [13] and Maurice de Gosson has explicitly addressed the matter of removing the hypothesis altogether in [3]. As these technical matters do not have a direct bearing on what we are up to it is not necessary to delve into them here.

On the other hand, it is proper to take note of two properties of τ, proven in [13], that foreshadow its algebraic role, particularly its connection to low-dimensional cohomology.

Proposition 1. *The Kashiwara-Maslov index τ satisfies the following properties:*

i. For all $\sigma \in Sp(V,B) = Sp(2n,\mathbb{R})$, $\tau(l_1^\sigma, l_2^\sigma, l_3^\sigma) = \tau(l_1, l_2, l_3)$.

ii. For all suitable $l_1, l_2, l_3, l_4 \in \mathfrak{Lag}(V)$, $\tau(l_1, l_2, l_3) = \tau(l_1, l_2, l_4) + \tau(l_1, l_3, l_4) + \tau(l_3, l_1, l_4)$, engendering a chain condition.

Next, returning to (3.16) and varying i, j through $\{1,2,3\}$ in all possible ways while suppressing x's, we get

$$W(l_i) = \mathsf{FT}_{i,j} \circ W(l_j) \circ \mathsf{FT}_{j,i} \tag{4.3}$$

whence, using the fact (cf. [13], p.34) that the indicated bounded unitary operator on $\mathcal{H}(l_1)$ commutes with the $W(l_i)(x)$ for all x, it follows essentially by an induction on n (see p.25 of *loc.cit.*) that there exists a unimodular scalar $a(l_1, l_2, l_3) \in \mathbb{C}_1^\times$ such that

$$a(l_1, l_2, l_3) = e^{\frac{i\pi}{4}\tau(l_1, l_2, l_3)}. \tag{4.4}$$

Furthermore, Lion and Vergne show that it is possible to pick a Lagrangian plane l_0 in V such that for all $\sigma_1, \sigma_2 \in Sp(V,B)$,

$$a(l_0, l_0^{\sigma_1}, l_0^{\sigma_2\sigma_1}) = c(\sigma_1, \sigma_2), \tag{4.5}$$

i.e.,

$$c(\sigma_1, \sigma_2) = e^{\frac{i\pi}{4}\tau(l_0, l_0^{\sigma_1}, l_0^{\sigma_2\sigma_1})}. \tag{4.6}$$

Thus, in (4.6) we have a direct relationship between the Weil 2-cocyle and the Kashiwara-Maslov index. The next order of business is to bring the Arnol'd-Leray-Maslov index into play.

The focus falls on the variety $\mathfrak{Lag}(V)$ as a topological space and its universal covering space $\widetilde{\mathfrak{Lag}}(V)$. As per, e.g., [1], [12], or [20], $\mathfrak{Lag}(V)$ can be identified with $U(n)/O(n)$, where $V \approx \mathbb{R}^{2n}$, whence, since $U(n)$'s covering space is

$$\widetilde{U(n)} = \{(A, \vartheta) \mid A \in U(n), \vartheta \in \mathbb{R} \text{ with } \det(A) = e^{i\vartheta}\}, \tag{4.7}$$

a Lie group, $\widetilde{\mathfrak{Lag}}(V)$ can be identified with $\widetilde{U(n)}/O(n)$, up to a diffeomorpism. What this means, of course, is that a point of $\widetilde{\mathfrak{Lag}}(V)$ can be realized as a pair $\widetilde{l} = (A, \vartheta)$ where A is unitary and $\vartheta \in \mathbb{R}$ is characterized by $e^{i\vartheta} = \det(\widetilde{l}) = \det(A)$, modulo $O(n)$. This allows us to synopsize the foregoing as

$$\widetilde{\mathfrak{Lag}}(V) = \{(A_l, \vartheta) \mid l \in \mathfrak{Lag}(V) \text{ and } \vartheta \in \mathbb{R} \text{ with } \det(A_l) = e^{i\vartheta}\} \tag{4.8}$$

where $\widetilde{l} = (A_l, \vartheta)$ sits above $l \in \mathfrak{Lag}(V)$ given the identification of l with a unitary matrix A_l in the presence of the aforementioned identification $\mathfrak{Lag}(V) \approx U(n)/O(n)$. The latter identification is made explicit by first providing an orthonormal basis $\{e_1, e_2, ..., e_3\}$ for l realized as a vector space (not over $\mathbb{R}$ but) over $\mathbb{C}$ by introducing a natural complex structure J (with $J^2 = -I$) on (V, B), preserving the symplectic form B, and equipping it with an inner product in the form of $-B(Jx, y) - iB(x, y)$ (cf. [20], p.122); subsequently A_l is just a column matrix: $A_l = (e_1, e_2, ..., e_3)$.

Now, if $\widetilde{l_1} = (A_1, \vartheta_1)$ and $\widetilde{l_2} = (A_2, \vartheta_2)$ in $\widetilde{\mathfrak{Lag}}(V)$ in accord with (4.8), then Souriau defines their Maslov index as

$$m(\widetilde{l_1}, \widetilde{l_2}) := \frac{1}{2\pi}\{\vartheta_1 - \vartheta_2 + iTrace(Log(-A_1 A_2^{-1}))\} \tag{4.9}$$

where, by definition (*loc.cit.*, p. 126),

$$Log(A) = \int_{\infty}^{0} \{(sI - A)^{-1} - (sI - I)^{-1}\} ds \tag{4.10}$$

for A a square matrix.

Next, turning to [2] by Cappell-Lee-Miller, let $\mathcal{P}([a, b], \mathfrak{Lag}(V)^2)$ be the set of continuous piece-wise smooth paths $f : [a, b] \to \mathfrak{Lag}(V) \times \mathfrak{Lag}(V)$ given parameterically as $f(t) = (l_1(t), l_2(t)), a \leq t \leq b$, so that $l_1(t)$ and $l_2(t)$ can regarded as piece-wise smoothly varying Largangian planes in V. Here the status of $\mathfrak{Lag}(V)$ as V's Lagrangian Grassmannian manifold plays a role: $\mathcal{P}([a, b], \mathfrak{Lag}(V)^2)$ becomes a topological space when endowed with the piece-wise smooth topology.

Under these circumstances an integer-valued mapping

$$\mu : \mathcal{P}([a, b], \mathfrak{Lag}(V)^2) \to \mathbb{Z} \tag{4.11}$$

is a (or, as we shall see, "the") Maslov index for (V, B) if it obeys the following six axioms which we include here for completeness and motivation:

Axiom 1. *For fixed $k > 0, l \geq 0$ in $\mathbb{R}$, let $\psi : [a.b] \to [ka + l, kb + l]$, so that composition with ψ maps $\mathcal{P}([ka + l, kb + l], \mathfrak{Lag}(V)^2)$ to $\mathcal{P}([a, b], \mathfrak{Lag}(V)^2)$. Then we require that*

$$\mu(f \circ \psi) = \mu(f). \tag{4.12}$$

Axiom 2. *Suppose that for all $s \in [0, 1]$, $f(s)(t) \in \mathcal{P}([a, b], \mathfrak{Lag}(V)^2)$ such that $f(s)$ is continuous on $[0, 1]$. Suppose, too, that for all $s_1, s_2 \in [0, 1]$, if $f(s)(t)$ is given as $(l_1(s)(t), l_2(s)(t))$, then $l_1(s_1)(a) = l_1(s_2)(a)$ and $l_2(s_1)(a) = l_2(s_2)(a)$, and also $l_1(s_1)(b) = l_1(s_2)(b)$ and $l_2(s_1)(b) = l_2(s_2)(b)$. Then*

$$\mu(f(0)) = \mu(f(1)). \tag{4.13}$$

Axiom 3. *If $f \in \mathcal{P}([a, b], \mathfrak{Lag}(V)^2)$ and $a < c < b$, then*

$$\mu(f) = \mu(f|_{[a,c]}) + \mu(f|_{[c,b]}). \tag{4.14}$$

Axiom 4. *Defining $\mathcal{P}([a,b],\mathfrak{Lag}(V\oplus W)^2)$ in the obvious manner, with W being a suitable symplectic space, too, we require that*

$$\mu(f\oplus g)=\mu(f)+\mu(g), \tag{4.15}$$

for $f\oplus g\in\mathcal{P}([a,b],\mathfrak{Lag}(V\oplus W)^2)$ in view of the stipulation that $f\in\mathcal{P}([a,b],\mathfrak{Lag}(V)^2$ and $g\in\mathcal{P}([a,b],\mathfrak{Lag}(W)^2)$.

Axiom 5. *Let $\varphi_\tau: V\xrightarrow{\sim}V$, i.e. $\varphi_\tau\in Sp(V)$, with τ varying continuously and piece-wise smoothly, and consider the pull-back*

$$\varphi_*(f)=\varphi_*(l_1(t),l_2(t)):[a,b]\to\mathfrak{Lag}(V)\times\mathfrak{Lag}(V) \tag{4.16}$$

given by

$$\varphi_*(f)(\tau)=\varphi_\tau(l_1(t),l_2(t)); \tag{4.17}$$

then

$$\mu(\varphi_*(f))=\mu(f). \tag{4.18}$$

Axiom 6. *Impart to $\mathbb{C}\approx\mathbb{R}^2$ the symplectic structure given by the rule $B(z_1,z_2)=B((x_1,y_1),(x_2,y_2))=x_1y_2-x_2y_1$ and consider the path $f_0\in\mathcal{P}([-\frac{\pi}{4},\frac{\pi}{4}],\mathfrak{Lag}(\mathbb{C})^2)$ given by $f_0(t)=(\mathbb{R}\cdot 1,\mathbb{R}\cdot e^{it})$. Then*

$$\mu(f_0)=1,\ \ \mu(f_0|_{[-\frac{\pi}{4},0]})=0,\ \ \textit{and}\ \mu(f_0|_{[0,\frac{\pi}{4}]})=1. \tag{4.19}$$

In the presence of these axioms the authors present the following centrally important result:

Proposition 2. *There exists one, and only one, mapping μ, as per (4.11), satisfying all six of these axioms. Furthermore, if $\xi:\mathcal{P}([a,b],\mathfrak{Lag}(V)^2)\to\mathbb{Z}$ satisfies Axioms 1-5, then there exist fixed integers A,B such that*

$$\xi(f)=(A+B)\mu(f)+B\{\dim(l_1(b)\cap l_2(b))-\dim(l_1(a)\cap l_2(a))\} \tag{4.20}$$

where, as before, $f(t)=(l_1(t),l_2(t))$, $a\le t\le b$.

For the proof, consult section 4 of [ ].

Under this regime, it follows that if $\pi:\widetilde{\mathfrak{Lag}}(V)\to\mathfrak{Lag}(V)$ is the natural projection map from its universal covering space to the Lagrangian Grassmannian of V, then

$$\tau(\pi(\widetilde{l}_1),\pi(\widetilde{l}_2,\pi(\widetilde{l}_3))=2\{m(\widetilde{l}_1,\widetilde{l}_2)+m(\widetilde{l}_2,\widetilde{l}_3)+m(\widetilde{l}_3,\widetilde{l}_1)\}. \tag{4.21}$$

This result, which is obviously central to our purposes, is ultimately a consequence of several theorems stated by Cappell, Lee, and Miller in [ ]; see especially their p. 172. However, a quicker presentation of this material is given by Maurice de Gosson in [4] (cf. p. 2, p.5), whose work along these lines is specifically cited by Cappell-Lee-Miller.

Now, having reached (4.21), we immediately obtain from (4.6) that, for any $\sigma_1, \sigma_2 \in Sp(V)$

$$c(\sigma_1, \sigma_2) = e^{i\frac{\pi}{2}\{m(\widetilde{l}_0, \widetilde{l}_0^{\sigma_1}) + m(\widetilde{l}_0^{\sigma_1}, \widetilde{l}_0^{\sigma_2\sigma_1}) + m(\widetilde{l}_0^{\sigma_2\sigma_1}, \widetilde{l}_0)\}} \tag{4.22}$$

by setting $l_1 = l_0, l_2 = l_0^{\sigma_1}, l_3 = l_0^{\sigma_2\sigma_1}$ and just noting that, generally, if $\widetilde{l} \in \widetilde{\mathfrak{Lag}}(V)$ is situated above $l \in \mathfrak{Lag}(V)$ then (tautologically) $\pi(\widetilde{l}) = l$.

It is this relation that makes for the possibility of forging a direct connection with Feynman's formalism for quantum mechanics, as we shall see in the upcoming section.

5. A certain Feynman integral

Jean-Marie Souriau's 1975 article, "Construction explicite de l'indice de Maslov. Applications" [20], conbtains in its tenth section, on the harmonic oscillator, a discussion of the indicated simple quantum mechanical system in terms of a Feynman integral (and thus a prototype of an oscillatory integral) and the Arnol'd-Leray-Maslov index m discussed above. Starting with a Lagrangian formalism in which the potential energy is given by the positive quadratic form $\frac{M}{2}\sum_{k=1}^{n}\omega_k^2 q_k^2$, where, working in n dimensions, $\overrightarrow{q} = (q_1, \cdots, q_n) \in \mathbb{R}^n$ and the $\omega_k, 1 \le k \le n$, provide the oscillator's proper periods in the form $\frac{2\pi}{\omega_k}$; of course M stands for mass. Souriau shown that if a_τ is the diagonal matrix $(e^{-i\omega_k\tau}\delta_{k,l})_{1\le k,l\le n}$, with $\delta_{k,l}$ the Kronecker delta and τ indicating the passage of time, then

$$m(u_t, u_{t+\tau}) = \frac{1}{2\pi}\{2(\omega_1 + \cdots + \omega_k) + iTrace(Log(-a_{2\tau}))\} \tag{5.1}$$

realizes the aforementioned index, $u_t, u_{t+\tau}$ being suitable Lagrangian planes; in this connection, compare (4.9). Furthermore, writing "Ent" for the greatest integer function, Souriau goes on to establish that

$$m(u_t, u_{t+\tau}) = \frac{n}{2} + \sum_{k=1}^{n} Ent(\frac{\omega_k\tau}{\pi}). \tag{5.2}$$

Then, with this regime in place, Souriau presents the following all-important result:

Proposition 3. *For $\overrightarrow{q}, \overrightarrow{q'} \in \mathbb{R}^n$, regarded as position vectors in a quantum mechanical phase space, and for ψ_t a state at time t (in a suitable Hilbert space),*

$$\psi_{t+\tau}(\overrightarrow{q}) = \{\prod_{k=1}^{n}\frac{\omega_k}{2\pi}|\csc(\omega_k\tau)|\}^{\frac{1}{2}} e^{i\frac{\pi}{2}m(u_t,u_{t+\tau})}\int_{\mathbb{R}^n}\psi_t(\overrightarrow{q'})e^{\frac{i}{2}S_{sou}(\overrightarrow{q},\overrightarrow{q'};\overrightarrow{\omega},\tau)}d\overrightarrow{q'} \tag{5.3}$$

where

$$S_{sou}(\overrightarrow{q}, \overrightarrow{q'}; \overrightarrow{\omega}, \tau) = \sum_{k=1}^{n}\omega_k\csc(\omega_k\tau)\{2q_kq'_k - (q_k^2 + (q'_k)^2)\cos(\omega_k\tau)\}, \tag{5.4}$$

$\overrightarrow{dq'} = dq'_1 dq'_2 \cdots dq'_n$, *and* τ *is not a half-period for the aforementioned harmonic oscillator.*

For the demonstration, see *loc.cit.*, p. 142, ff.

Manifestly the integral in (5.3) fits into Feynman's scheme for quantum mechanics and, as we shall see in our next section, is the final link in the chain as far as our staed objective is concerned. However, before we carry out this last step in our argument we take note of the fact that Souriau's result is closely related to a more recent result due to Robbin and Salamon: in [17] (cf. their Theorem 8.5) they present the relation

$$U(t_1, t_0; H)(f(\vec{x})) = \frac{e^{i\frac{\pi}{2}\mu(t_0,t_1;H)}}{(2\pi)^{\frac{n}{2}}|\det B|^{\frac{1}{2}}} \int_{\mathbb{R}^n} f(\vec{x'}) e^{iS(\vec{x},\vec{x'})} \overrightarrow{dx'} \tag{5.5}$$

where $\vec{x}, \vec{x'} \in \mathbb{R}^n$, of course, $U(t_1, t_0; H)$ is an evolution operator (from time t_0 to time t_1) and a generating function for a symplectomorphism $\psi_{t_0}^{t_1}$, H is a (quadratic) Hamiltonian, B is a time-dependent square matrix that is part of the decomposition of $\psi_{t_0}^{t_1}$, $S(\vec{x}, \vec{x'})$ is the attendant action (or phase), and, indeed, $\mu(t_0, t_1; H)$ is a Maslov index. Thus, we have in $\psi_{t_0}^{t_1}$ a change of state from time time t_0 to time t_1, and the unitary operator $U(t_1, t_0; H)$ conveys this evolution: we are on familiar quantum mechanical ground, and if we identify the phase $S(\vec{x}, \vec{x'})$ with the earlier action $\frac{1}{2}S_{sou}(\vec{q}, \vec{q'}; \vec{\omega}, \tau)$ coming from (5.3), then it appears that (5.5) coveys essentially the same information as (5.3). Thus, with a little more work (which we leave as an exercise for the reader) it should be possible to tailer what we do here to the setting considered by Robbin and Salamon.

6. From Weil to Feynman

With Proposition 3 in place, specifically (5.3), and with (4.22) available, we are at last in a position to write down the direct connection between Weil's metaplectic group's defining 2-cocycle and the indicated facets of quantum mechanics Feynman's idiom. To wit:

Proposition 4. *Write* $\vec{\omega}_0^{\sigma_1}, \vec{\omega}_{\sigma_1}^{\sigma_2\sigma_1}, \vec{\omega}_{\sigma_2\sigma_1}^{0}$ *for the data* $\vec{\omega} = (\omega_1, \cdots, \omega_n)$ *corresponding to, respectively,* $m(\tilde{l}_0, \tilde{l}_0^{\sigma_1}), m(\tilde{l}_0^{\sigma_1}, \tilde{l}_0^{\sigma_2\sigma_1}), m(\tilde{l}_0^{\sigma_2\sigma_1}, \tilde{l}_0)$, *taking* $u_0 = \tilde{l}_0$, $u_{\tau_{\sigma_1}} = \tilde{l}_0^{\sigma_1}$; $u_0 = \tilde{l}_0^{\sigma_1}$, $u_{\tau_{\sigma_2\sigma_1}} = \tilde{l}_0^{\sigma_2\sigma_1}$; $u_0 = \tilde{l}_0^{\sigma_2\sigma_1}$, $u_\tau = \tilde{l}_0$, *again respectively. Then, for all* $\sigma_1, \sigma_2 \in Sp(V)$,

$$\begin{aligned} c(\sigma_1, \sigma_2) \;=\; & \frac{\{\prod_{k=1}^n (\vec{\omega}_0^{\sigma_1})_k (\vec{\omega}_{\sigma_1}^{\sigma_2\sigma_1})_k (\vec{\omega}_{\sigma_2\sigma_1}^{0})k|\csc((\vec{\omega}_0^{\sigma_1})_k\tau_{\sigma_1})\csc((\vec{\omega}_{\sigma_1}^{\sigma_2\sigma_1})_k\tau_{\sigma_2\sigma_1})\csc((\vec{\omega}_{\sigma_2\sigma_1}^{0})_k\tau|\}^{\frac{1}{2}}}{(8\pi^3)^{\frac{n}{2}}\psi_{\tau_{\sigma_1}}(\vec{q}_1)\varphi_{\tau_{\sigma_2\sigma_1}}(\vec{q}_2)\xi_\tau(\vec{q}_3)} \times \\ & \times \int_{\mathbb{R}^{3n}} \psi_0(\vec{q'_1})\varphi_0(\vec{q}'_2)\xi_0(\vec{q}'_3) e^{\frac{i}{2}\{S_{sou}(\vec{q_1}\vec{q'_1},;\vec{\omega}_0^{\sigma_1},\tau_{\sigma_1}) + S_{sou}(\vec{q_2}\vec{q'_2},\vec{\omega}_{\sigma_1}^{\sigma_2\sigma_1},\tau_{\sigma_2\sigma_1}) + S_{sou}(\vec{q_3},\vec{q'_3};\vec{\omega}_{\sigma_2\sigma_1}^{0},\tau)\}} \overrightarrow{dq'_1}\, d\vec{q}'_2\, d\vec{q}'_3 \end{aligned} \tag{6.1}$$

where $\vec{q}_j, \vec{q}'_j$, $j = 1, 2, 3$, range over $\mathbb{R}^n$ and the pairs $\psi_{\tau_{\sigma_1}}, \psi_0; \varphi_{\tau_{\sigma_2\sigma_1}}, \varphi_0; \xi_\tau, \xi_0$ are the indicated instances of $\psi_{t+\tau}, \psi_t (t = 0)$ as given in (5.3) (which is to say that we have opted for φ, ξ to mitigate an even more cumbersome expression: these are just the appropriate ψ's in (5.3)).

Proof: With the stated assignment in place, solve (5.3) for $e^{\frac{i\pi}{2} m(u_0, u_\tau)}$, letting τ take on the values given above. Then the result follows directly from (4.22) plus an obvious application of Fubini's Theorem.

For notational convenience we can restate our result more compactly as follows, using the usual multi-index conventions and some obvious abbreviations:

$$\begin{aligned} c(\sigma_1, \sigma_2) &= (8\pi^3)^{\frac{-n}{2}} \frac{\{\vec{\omega}_0^{\sigma_1} \vec{\omega}_{\sigma_1}^{\sigma_2\sigma_1} \vec{\omega}_{\sigma_2\sigma_1}^{0} | \csc(\vec{\omega}_0^{\sigma_1} \tau_{\sigma_1}) \csc(\vec{\omega}_{\sigma_1}^{\sigma_2\sigma_1} \tau_{\sigma_2\sigma_1}) \csc(\vec{\omega}_{\sigma_2\sigma_1}^{0} \tau) |\}^{\frac{1}{4}}}{\psi_{\tau_{\sigma_1}}(\vec{q}_1) \varphi_{\tau_{\sigma_2\sigma_1}}(\vec{q}_2) \xi_\tau(\vec{q}_3)} \times \\ &\quad \times \int_{\mathbb{R}^{3n}} \psi_0(\vec{q}'_1) \varphi_0(\vec{q}'_2) \xi_0(\vec{q}'_3) e^{\frac{i}{2}(S^{\sigma_1} + S^{\sigma_2\sigma_1} + S^0)} d\vec{q}'_1 d\vec{q}'_2 d\vec{q}'_3. \end{aligned}$$

(6.2)

In view of our remarks at the close of the preceding section and the theorems of Cappell-Lee-Miller discussed above, our relations (6.1), (6.2) should be easily amenable to translation into the language of Robbin-Salamon (cf. (5.5)), setting the stage for an interpretation of the foregoing in terms of oscillatory integrals, but, in any event, we have in these identities, i.e. in Proposition 4, a direct formulaic connection between Weil's 2-cocycle $c \in H^2(Sp(V), \mathbb{Z})$ and a Feynman integral. Turning the tables, this relation should also make for an evaluation scheme for the indicated Feynman integrals in terms of the 2-values 2-cocycle: $c = \pm 1$ after all, and this is promising in its own right.

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